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On n-absorbing ideals and (m, n)-closed ideals in trivial ring extensions of commutative rings

Ayman Badawi*,‡, Mohammed Issoual†,§ and Najib Mahdou†,¶

*Department of Mathematics and Statistics
The American University of Sharjah, P. O. Box 26666
Sharjah, United Arab Emirates

†Department of Mathematics
Faculty of Science and Technology
University S. M. Ben Abdellah, Fez 30000, Morocco
‡abadawi@aus.edu
§issoual2@yahoo.fr
¶mahdou@hotmail.com

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Let R be a commutative ring with $1 \neq 0$. Recall that a proper ideal I of R is called a 2-*absorbing ideal* of R if $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. A more general concept than 2-absorbing ideals is the concept of n-absorbing ideals. Let $n \geq 1$ be a positive integer. A proper ideal I of R is called an n-*absorbing ideal* of R if $a_1, a_2, \ldots, a_{n+1} \in R$ and $a_1, a_2 \cdots a_{n+1} \in I$, then there are n of the a_i's whose product is in I. The concept of n-absorbing ideals is a generalization of the concept of prime ideals (note that a prime ideal of R is a 1-absorbing ideal of R). Let m and n be integers with $1 \leq n < m$. A proper ideal I of R is called an (m, n)-*closed ideal* of R if whenever $a^m \in I$ for some $a \in R$ implies $a^n \in I$. Let A be a commutative ring with $1 \neq 0$ and M be an A-module. In this paper, we study n-absorbing ideals and (m, n)-closed ideals in the trivial ring extension of A by M (or idealization of M over A) that is denoted by $A(+)M$.

Keywords: Prime ideal; radical ideal; 2-absorbing ideal; n-absorbing ideal; (m, n)-closed ideal; trivial extension; idealization of a ring.

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1. Introduction

We assume throughout that all rings are commutative with $1 \neq 0$. Over the past several years, there has been considerable attention in the literature to n-absorbing ideals of commutative rings and their generalizations, for example see ([2–8, 10–22,

¶Corresponding author.

24–29, 31]). We recall from [4] that a proper ideal I of R is called a *2-absorbing ideal* of R if $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. A more general concept than 2-absorbing ideals is the concept of n-absorbing ideals. Let $n \geq 1$ be a positive integer. A proper ideal I of R is called an *n-absorbing ideal* of R as in [2] if $a_1, a_2, \ldots, a_{n+1} \in R$ and $a_1, a_2 \cdots a_{n+1} \in I$, then there are n of the a_i's whose product is in I. A proper ideal of R is called a *strongly n-absorbing ideal* of R as in [2] if whenever $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R, then the product of some n of the I_j's is contained in I. The concept of n-absorbing ideals is a generalization of the concept of prime ideals (note that a prime ideal of R is a 1-absorbing ideal of R). Let m and n be the positive integers with $1 \leq n < m$. We recall from [3] that a proper ideal I of R is called an *(m, n)-closed ideal* of R if whenever $a^m \in I$ for some $a \in I$ implies $a^n \in I$.

Let A be a commutative ring and M be an A-module. *The trivial ring extension of A by M* (or the idealization of M over A) is the ring $R = A(+)M$ whose underlying group is $A \times M$ with multiplication given by $(a, b)(c, d) = (ac, ad + bc)$ (for example see [23]). In this paper, we study n-absorbing ideals, strongly n-absorbing ideals, and (m, n)-closed ideals in the ring $R = A(+)M$. We start by recalling some background materials. We say A is a *quasilocal* ring if A has exactly one maximal ideal. If I is a primary ideal of a ring A with $\sqrt{I} = P$ (a prime ideal of A), then we say that I is a *P-primary* ideal of A. A prime ideal P of a ring A is called *divided* if $P \subset x$ for every $x \in A \backslash P$. Suppose that I is a n-absorbing ideal of a ring A for some integer $n \geq 1$. Then, as in [2], we put $w_A(I) = \min\{n \in \mathbb{N} \,|\, I \text{ is } n\text{-absorbing ideal of A}\}$, and $w_A^*(I) = \min\{n \in \mathbb{N} \,|\, I \ \text{ is a strongly } n\text{-absorbing ideal of A}\}$. Let A be a commutative ring and M be an A-module. Then a submodule N of M is called a *P-primary submodule* of M for some prime ideal P of A if $(N : M) = \{x \in A \,|\, xM \subseteq N\}$ is a primary ideal of A with $\sqrt{(N : M)} = \{a \in A \,|\, a^n M \subseteq N \text{ for some integer } n \geq 1\} = P$.

Let $n \geq 1$ be an integer and I be a proper ideal of A. Anderson and Badawi in [2] (also see [10]) proposed the following three conjectures:

(1) Conjecture one: I is an n-absorbing ideal of A if and only if I is a strongly n-absorbing ideal of A.
(2) Conjecture two: If I is an n-absorbing ideal of A, then $(\sqrt{I})^n \subseteq I$. An affirmative answer to this conjecture is given in [15].
(3) Conjecture three: If I is an n-absorbing ideal of A, then $I[X]$ is an n-absorbing ideal of $A[X]$.

In this paper, we study the validity of the above three conjectures in the ring $R = A(+)M$.

2. n-Absorbing Ideals in Trivial Ring Extensions

We recall [1, Corollary 3.4] that if A is an integral domain and M is a divisible A-module, then every ideal of $A(+)M$ has the form $I(+)M$ for some proper ideal I of A or $0(+)N$ for some submodule N of M.

In the following result, we collect some trivial facts about n-absorbing ideals and (m, n)-closed ideals in $R = A(+)M$ and hence we omit the proof.

Theorem 2.1. *Let A be a commutative ring, I be a proper ideal of A, M be an A-module, and $R = A(+)M$. Then*

(1) *I is an n-absorbing ideal of A if and only if $I(+)M$ is an n-absorbing ideal of R.*
(2) *I is a strongly n-absorbing ideal of A if and only if If $I(+)M$ is a strongly n-absorbing of R.*
(3) *I is an (m, n)-closed ideal of A if and only if $I(+)M$ is an (m, n)-closed ideal of R.*

Example 2.2. Let A be a field and M be an A-vector space. It is clear that $R = A(+)M$ is a quasilocal ring with the maximal is $M = \{0\}(+)M$. Since $M^2 = \{0\}$, we conclude that every ideal of R is a 2-absorbing ideal of R and hence a strongly 2-absorbing ideal of R by [4, Theorem 2.13]. Thus every ideal of R is a strongly n-absorbing ideal of R for every $n \geq 2$.

We recall the following results.

Theorem 2.3. (1) ([15]) *If I is an n-absorbing ideal of a ring A for some integer $n \geq 1$, then $(\sqrt{I})^n \subseteq I$.*

(2) ([2, Theorem 3.1]) *Let P be a prime ideal of a ring A, and let I be a P-primary ideal of A such that $P^n \subseteq I$ for some positive integer n (for example, if A is a Noetherian ring). Then I is an n-absorbing ideal of A.*
(3) ([2, Theorem 6.6]) *Let P be a prime ideal of a ring A, I be a P-primary ideal of A, and $n \geq 1$ be an integer. Then I is a strongly n-absorbing ideal of A if and only if $P^n \subseteq I$ and I is an n-absorbing ideal of R.*
(4) ([2, Theorem 3.2]) *Let P be a divided prime ideal of A, and let I be an n-absorbing ideal of A with $\sqrt{I} = P$. Then I is a P-primary ideal of A.*
(5) ([2, Theorem 3.3]) *Assume that $\sqrt{\{0\}} \subset P$ are divided prime ideals of A and $n \geq 1$ be an integer. Then P^n is a P-primary ideal of A, and thus P^n is an n-absorbing ideal of A.*

In view of Theorem 2.3, we have the following result.

Corollary 2.4. (1) *Let P be a prime ideal of a ring A, $n \geq 1$ be an integer, and let I be a P-primary ideal of A. Then I is an n-absorbing ideal of A if and only if $P^n \subseteq I$ if and only if I is a strongly n-absorbing ideal of A.*

(2) *Let P be a divided prime ideal of A, and let I be a proper ideal of A with $\sqrt{I} = P$. Then I is an n-absorbing ideal of A if and only if I is a P-primary ideal of A and $P^n \subseteq I$ if and only if I is a strongly n-absorbing ideal of A.*
(3) *Assume that $\sqrt{\{0\}} \subset P$ are divided prime ideals of A and $n \geq 1$ be an integer. Then P^n is a strongly n-absorbing ideal of A.*

Proof. (1) By Theorem 2.3[(1), (2), (3)], the claim follows.
(2) By Theorem 2.3[(4), (1), (2), (3)], the claim follows.
(3) By Theorem 2.3[(5), (2), (3)], the claim follows. □

Theorem 2.5. *Let A be a commutative ring, M be an A-module, $R = A(+)M$, $n \geq 1$ be an integer, I be a proper ideal of A, and N be a submodule of M such that $IM \subseteq N$. Then:*

(1) *If $I(+)N$ is an n-absorbing ideal of R, then I is an n-absorbing ideal of A.*
(2) *Let P be a prime ideal of A, I be a P-primary ideal of A, and N be a P-primary submodule of M. Then I is an n-absorbing ideal of A if and only if $I(+)N$ is an n-absorbing ideal of R.*
(3) *Let P be a prime ideal of A, I be a P-primary ideal of A, and N be a P-primary submodule of M. Then $I(+)N$ is an n-absorbing ideal of R if and only if $I(+)N$ is a strongly n-absorbing ideal of R.*
(4) *Let P be a divided prime ideal of A, I be an n-absorbing ideal of A with $\sqrt{I} = P$, and N be a P-primary submodule of M. Then $I(+)N$ is a strongly n-absorbing ideal of R.*
(5) *Assume that $\sqrt{\{0\}} \subset P$ are divided prime ideals of A such that $P^n M \subseteq N$. If N is a P-primary submodule of M, then $P^n(+)N$ is a strongly n-absorbing ideal of R.*
(6) *Assume that A is a Prüfer domain and let $J = I(+)M$. Then $J = I(+)M$ is an n-absorbing ideal of R if and only if J is a strongly n-absorbing ideal of R. Moreover $w(J) = w^*(J)$.*

Proof. (1) No comments.

(2) Since I is a P-primary ideal of A and N is a P-primary submodule of M, we conclude that $I(+)N$ is a $P(+)M$-primary ideal of R by [1, Theorem 3.6]. Suppose that I is an n-absorbing ideal of A. Then $(\sqrt{I})^n = P^n \subseteq I$ by Theorem 2.3(1). Hence $(\sqrt{I(+)N})^n = (P(+)M)^n \subseteq P^n(+)N \subseteq I(+)N$. Thus, $I(+)N$ is an n-absorbing ideal of R by Corollary 2.4(1). Conversely, suppose that $I(+)N$ is an n-absorbing ideal of R. Then $(\sqrt{I(+)N})^n = (P(+)M)^n \subseteq I(+)N$ by Theorem 2.3(1). In particular, $P^n \subseteq I$. Since I is a P-primary ideal of A and $P^n \subseteq P$, we conclude that I is an n-absorbing ideal of A by Corollary 2.4(1).
(3) Since $I(+)N$ is a $P(+)M$-primary ideal of R by [1, Theorem 3.6] and $(\sqrt{I(+)N})^n = (P(+)M)^n \subseteq I(+)N$ by Theorem 2.3(1), the claim follows by Theorem 2.3(3).
(4) By Corollary 2.4(2), we conclude that I is a P-primary ideal of A. Hence we are done by (2) and (3).
(5) By Theorem 2.3, we conclude that P^n is a P-primary ideal of A and hence an n-absorbing ideal of A. Thus we are done by (2) and (3).
(6) Suppose that $J = I(+)M$ is an n-absorbing ideal of R. Then I is an n-absorbing ideal of A. Since A is a Prüfer domain, we conclude that I is a strongly

n-absorbing ideal of A by [2, Corollary 6.9]. Hence $J = I(+)M$ is s strongly n-absorbing ideal of R. The converse is clear. It is clear that $w(J) = w^*(J)$. □

3. Conjecture One in Trivial Ring Extension

Let $n \geq 1$ be an integer and I be a proper ideal of a ring A. Anderson and Badawi in [2] (also see [10]) proposed the following conjecture.

Conjecture one: I is an n-absorbing ideal of A if and only if I is a strongly n-absorbing ideal of A.

Laradji in [27] proved that conjecture one holds in some rings that satisfy certain conditions. In particular, he proved that Conjecture three implies Conjecture one. We have the following lemma.

Lemma 3.1. *Let A be an integral domain with quotient field K, M be a K-vector space, F be a K-subspace of M, and $R = A(+)M$. Then $J = \{0\}(+)F$ is a strongly 2-absorbing ideal of R, and thus J is a strongly n-absorbing ideal of R for every $n \geq 2$.*

Proof. First, we show that J is a 2-absorbing ideal of R. Let $x_i = (a_i, e_i) \in R$, where $1 \leq i \leq 3$. Suppose that $x_1x_2x_3 \in \{0\}(+)F$. Since A is an integral domain, we may assume that $a_3 = 0$. Suppose that $a_1a_2 = 0$. Then $x_1x_3 \in J$ or $x_2x_3 \in J$. Suppose that $a_1a_2 \neq 0$. Then $x_1x_2x_3 = (0, a_1a_2e_3)$. Since F is a K-subspace of M, we conclude that $a_2^{-1}a_1^{-1}(a_1a_2e_3) = e_3 \in F$. Hence $x_3 = (0, e_3) \in J$, and thus $x_1x_3 \in J$. Hence J is a 2-absorbing ideal of R. Thus, J is a strongly 2-absorbing ideal of R by [4, Theorem 2.13], and hence J is a strongly n-absorbing ideal of R for every $n \geq 2$. □

Theorem 3.2. *Let A be an integral domain with quotient field K, M be a K-vector space, F be an A-submodule of M, and $R = A(+)M$. Then $\{0\}(+)F$ is an n-absorbing ideal of R for some $n \geq 2$ if and only if F is a K-subspace of M.*

Proof. Suppose that $J = \{0\}(+)F$ is an n-absorbing ideal of R for some $n \geq 2$. Let a be a nonzero element of A and $f \in F$. We show $\frac{1}{a}f \in F$. Let $x = (a, 0), y = (0, \frac{f}{a^n}) \in R$. Then $x^n y = (0, f) \in J$. Since $a \neq 0$ and J is an n-absorbing ideal of R, we conclude that $x^{n-1}y = (0, \frac{f}{a}) \in J$. Thus, $\frac{1}{a}f \in F$. Now let $h \in K$ and $v \in F$. Then $h = \frac{b}{c} \in K$ for some $b, c \in A$ with $c \neq 0$. Since $\frac{1}{c}v \in F$ and F is an A-submodule of M, we conclude that $hv = \frac{b}{c}v \in F$. Thus, F is a K-subspace of M. The converse is clear by Lemma 3.1. □

Corollary 3.3. *Let A be an integral domain that is not a field with quotient field K, and $R = A(+)K$. Then $J = \{0\}(+)A$ is not an n-absorbing ideal of R for every $n \geq 1$.*

Proof. Since A is not a field, we conclude that A is not a K-subspace of K. Hence we are done by Theorem 3.2. □

Theorem 3.4. *Let A be an integral domain with quotient field K, M be a K-vector space, and $R = A(+)M$. Then Conjecture one holds in R if and only if Conjecture one holds in A.*

Proof. First, observe that M is a divisible A-module. Hence every ideal of $R = A(+)M$ has the form $I(+)M$ for some proper ideal I of A or $0(+)N$ for some submodule N of M by [1, Corollary 3.4].

Suppose that Conjecture one holds in R. Let I be a proper n-absorbing ideal of A for some integer $n \geq 1$. Then $J = I(+)M$ is a n-absorbing ideal of $R = A(+)M$, and hence a strongly n-absorbing ideal R by hypothesis. Thus, I is a strongly n-absorbing ideal of A by Theorem 2.1(2).

Conversely, suppose that Conjecture one holds in A. Let J be a proper n-absorbing ideal of $R = A(+)M$ for some $n \geq 1$. Hence J is the form $I(+)M$ where I is a proper ideal of A or $0(+)F$ where F is a K-subspace of M.

Case 1. $J = I(+)M$, *where I is a proper ideal of A*. Since J is an n-absorbing ideal of R, we conclude that I is an n-absorbing ideal of A by Theorem 2.1(1), and hence I is a strongly n-absorbing ideal of A by hypothesis. Thus, $J = I(+)M$ is a strongly n-absorbing ideal of $R = A(+)M$ by Theorem 2.1(2).

Case 2. $J = \{0\}(+)F$, *where F is an A-submodule of M*. If $n = 1$, then $F = M$ and we are done. Hence assume that $n \geq 2$. Since J is an n-absorbing ideal of R, we conclude that F is a K-subspace of M by Theorem 3.2. Hence J is a strongly n-absorbing ideal of R for every $n \geq 2$ by Lemma 3.1. Thus, Conjecture one holds in $R = A(+)M$. □

Corollary 3.5. *Let A be a Prüfer domain with quotient field K, M be K-vector space, and $R = D(+)M$. Then Conjecture one holds in R.*

Proof. Since A is a Prüfer domain, Conjecture one holds in A by [2, Corollary 6.9]. Thus Conjecture one holds in R by Theorem 3.4. □

We recall the following result.

Theorem 3.6 ([2, Corollary 6.8]). *Let R be a Noetherian ring. Then every proper ideal of R is a strongly n-absorbing ideal of R for some positive integer n.*

Theorem 3.7. *Let A be a Noetherian ring, M be an A-module, $R = A(+)M$, and I be a proper ideal of A. Then $J = I(+)M$ is a strongly n-absorbing ideal of R for some positive integer n.*

Proof. Since I is a strongly n-absorbing ideal of A for some positive integer n by Theorem 3.6, we conclude that $J = I(+)M$ is a strongly n-absorbing ideal of R. □

Theorem 3.8. *Let A be a Noetherian ring, M be a finitely generated A-module, and $R = A(+)M$. Then every ideal of R is a strongly n-absorbing ideal of R for some positive integer n.*

Proof. Since A be a Noetherian ring and M is a finitely generated A-module, we conclude that R is a Noetherian ring by [1, Theorem 4.8]. Hence the claim follows from Theorem 3.6. □

Question 1. In view of Theorem 3.6, El Amin El Kaidi asked the following question: Let A be a ring and assume that every ideal of A is an n-absorbing ideal of R for some integer $n \geq 1$. Does it imply that A is a Noetherian ring?

The following example gives a negative answer to the above question.

Example 3.9. Let $A \subset K$ be fields such that K is not a finitely generated A-module (for example, let $A = \mathbb{Q}$ and $K = \mathbb{R}$) and $R = A(+)K$. Since R is a quasilocal ring with maximal ideal $M = \{0\}(+)K$ and $M^2 = \{(0,0)\}$, we conclude that every ideal of R a 2-absorbing ideal of R (and hence every ideal of R is a strongly n-absorbing ideal of R for every $n \geq 2$ by [4, Theorem 2.13]). Since K is not a finitely generated A-module, we conclude that $\{0\}(+)K$ is not a finitely generated of R. Thus R is not a Noetherian ring.

Remark 3.10. Let R be a ring and n a positive integer such that every proper ideal of R is an n-absorbing ideal of R. Then by [2, Theorem 5.9], we have $\dim(R) = 0$ and R has at most n maximal ideals.

We have the following result.

Theorem 3.11. *Let A be an integral domain with quotient field K, M be a finite dimensional vector space over K, and $R = A(+)M$. Then every proper ideal of R is an n-absorbing ideal of R for some $n \geq 1$ if and only if $A = K$.*

Proof. Suppose that $A = K$. Since M is a finite dimensional vector space over K, we conclude that R a Noetherian ring by [1, Theorem 4.8]. Hence every proper ideal of R is an n-absorbing ideal of R for some $n \geq 1$ by Theorem 3.6. Conversely, suppose that every proper ideal of R is an n-absorbing ideal of R for some $n \geq 1$. Since M is a finite dimensional vector space over K, we may assume that $M = K \times \cdots \times K$ (m times, where $m = \dim_K(M) < \infty$). Hence $N = A \times \cdots \times A$ is a an A-submodule of M and $J = \{0\} \times N$ is a 2-absorbing ideal of R. Since $J = \{0\} \times N$ is a 2-absorbing ideal of R, we conclude that N is a K-subspace of M by Theorem 3.2. Thus, $A = K$. □

In light of Theorems 3.6 and 3.11, we have the following result.

Corollary 3.12. *Let A be an integral domain with quotient field K, M be a finite dimensional vector space over K, and $R = A(+)M$. Then the following statements* *are equivalent.*

(1) *Every proper ideal of R is a strongly n-absorbing ideal of R for some $n \geq 1$.*
(2) *Every proper ideal of R is an n-absorbing ideal of R for some $n \geq 1$.*
(3) $A = K$.
(4) *A is a Noetherian ring.*
(5) *R is a Noetherian ring.*

Theorem 3.13. *Let A be a Noetherian domain with quotient field K, M be a K-vector space, and $R = A(+)M$. Then a proper ideal J of R is an n-absorbing ideal of R for some $n \geq 1$ if and only if J is a strongly m-absorbing ideal of R for some $m \geq 1$.*

Proof. If $n = 1$ or $m = 1$. Then J is a prime ideal of R, and hence the claim is clear. Let J be a proper ideal of R. Since M is a divisible A-module, we conclude that $J = I(+)M$ for some proper ideal I of A or $J = \{0\}(+)F$ for some A-submodule F of M by [1, Corollary 3.4]. Suppose that J is n-absorbing ideal of R for some $n \geq 2$. Assume that $J = I(+)M$ for some proper ideal I of A. Since I is a strongly m-absorbing ideal of A for some positive integer m by Theorem 3.6, we conclude that $J = I(+)M$ is a strongly m-absorbing ideal of R. Suppose that $J = \{0\}(+)F$ for some A-submodule F of M. Then F is a K-subspace of M by Theorem 3.2. Thus J is a strongly k-absorbing ideal of R for every integer $k \geq 2$ by Lemma 3.1. The converse is clear. □

4. Conjecture Three in Trivial Ring Extension

Let A be a commutative ring, and M an A-module, let $R = A(+)M$, we know $(A(+)M)[X]$ is naturally isomorphic to $A[X](+)M[X]$. If I is a ideal of A, then $(I(+)M)[X]$ is naturally isomorphic to $I[X](+)M[X]$.

We recall the following result.

Theorem 4.1 ([2, Theorem 4.15]). *Let I be a proper ideal of a ring A. Then $I[X]$ is a 2-absorbing ideal of $R[X]$ if and only if I is a 2-absorbing ideal of R.*

Theorem 4.2. *Let A be an integral domain with quotient field K, M be a K-vector space, and $R = A(+)M$. Then Conjecture three holds in R if and only if Conjecture three holds in A.*

Proof. Suppose the Conjecture three holds in A. Let J be a proper n-absorbing ideal of R for some $n \geq 1$. Hence $J = I(+)M$ for some proper ideal I of A or $J = \{0\}(+)F$ for some K-subspace F of M by [1, Corollary 3.4] and Theorem 3.2.

Case 1. Suppose that $J = I(+)M$ for some proper ideal I of A. Then I is an n-absorbing ideal of A. Thus $I[X]$ is an n-absorbing ideal of $A[X]$ by hypothesis. Hence $w_A(I) = w_{A[X]}(I[X])$. Since $J[X]$ is isomorphic to $I[X](+)M[X]$, we conclude that $J[X]$ is an n-absorbing ideal of $R[X]$. Since $w_{R[X]}(J[X]) = w_{A[X](+)M[X]}(I[X](+)M[X]) = w_{A[X]}(I[X]) = w_A(I)$. Hence $w_{R[X]}(J[X]) = w_R(J)$.

Case 2. Suppose that $J = 0(+)F$ for some K-subspace F of M.

Since J is a 2-absorbing ideal of R, we conclude that $J[X]$ is a 2-absorbing absorbing ideal of $R[X]$ by Theorem 4.1. Hence Conjecture three holds in R.

Conversely, suppose that Conjecture three holds in R. Let I be an n-absorbing ideal of A. Then $I(+)M$ is n-absorbing ideal of R. Hence $(I(+)M)[X]$ is n-absorbing ideal of $R[X]$ by hypothesis. Since $(I(+)M)[X] \cong I[X](+)M[X]$, we conclude that $I[X]$ is an n-absorbing ideal of $A[X]$. □

Laradji [27, Corollary 2.11] showed that Conjecture three holds in arithmetical rings. Since a Prüfer domain is both arithmetical and Gaussian ring, the following result is an immediate consequence of [27, Corollary 2.11] and [31, Theorem 2.6].

Lemma 4.3 ([27, Corollary 2.11] and [31, Theorem 2.6]). *Let A be a Prüfer domain and I be a proper n-absorbing ideal of A for some integer $n \geq 1$. Then $I[X]$ is an n-absorbing ideal of $A[X]$.*

In the following result, we construct rings with zero-divisors that satisfy Conjecture three but they do not need be arithmetical rings.

Theorem 4.4. *Let A be a Prüfer domain with quotient field K, M be K-vector space, n be a positive integer, and J be a proper ideal of $R = A(+)M$ (note that if $M = K[X]$, then R is not an arithmetical ring by [9, Theorem 2.1(2)]). If J is an n-absorbing ideal of R, then $J[X]$ is an n-absorbing ideal of $R[X]$ and $w_R(J) = w_{R[X]}(J[X])$.*

Proof. Since A is a Prüfer domain, Conjecture three holds in A by Lemma 4.3. Thus Conjecture three holds in R by Theorem 4.2. Thus, If J is an n-absorbing ideal of R, then $J[X]$ is an n-absorbing ideal of $R[X]$ and $w_R(J) = w_{R[X]}(J[X])$. □

In the following example, we construct a non-arithmetical ring that satisfies Conjecture three.

Example 4.5. Let A be a Prüfer domain with quotient field K, $M = K[X]$, and $R = A(+)M$. Then:

(1) R satisfies Conjecture three by Theorem 4.4.
(2) R is a non-arithmetical ring by [9, Theorem 2.1(2)].

Remark 4.6. Let I be a proper ideal of a ring A and $n \geq 1$. It is shown [2, Theorem 6.1] that if I is a strongly n-absorbing ideal of A, then $(\sqrt{I})^n \subseteq I$. It is shown [27, Proposition 2.9(1)] that if $I[X]$ is an n-absorbing ideal of $A[X]$, then I is a strongly n-absorbing ideal of A. It is shown [27, Corollary 2.11] that if I is an

n-absorbing ideal of an arithmetical ring A, then $I[X]$ is an n-absorbing ideal of $A[X]$. Hence if A is an arithmetical ring, then all three Conjectures hold in A.

In the following result, we construct rings with zero-divisors that satisfy all three conjectures but they do not need be arithmetical rings.

Theorem 4.7. *Let A be a Prüfer domain with quotient field K, M be K-vector space, n be a positive integer, and $R = A(+)M$ (note that if $M = K[X]$, then R is not an arithmetical ring by* [9, *Theorem* 2.1(2)]*). Suppose that J is an n-absorbing ideal of R. Then the following statements hold*:

(1) *J is a strongly n-absorbing ideal of R.*
(2) *$J[X]$ is an n-absorbing ideal of R.*
(3) $(\sqrt{J})^n \subseteq J$.

Proof. (1) It is clear by Corollary 3.5.
(2) It is clear by Theorem 4.4.
(3) It is clear by [15]. □

5. Conjecture One in u-Rings

We recall from [30] that commutative ring R is called a u-ring if whenever an ideal I of R is contained in a finite union of ideals of R, then I is contained in at least one of those ideals. It is known that every Bezout ring is a u-ring and every Prüfer domain is a u-domain. In [31, Theorem 2.4], Smach and Hizem showed that Conjecture one holds in u-rings. In this section, we propose a proof of this result that is different from that in [31, Theorem 2.4]. We need the following notation. Let R be a commutative ring. If $x_1, \dots, x_n \in R$, then $x_1, \dots, \widehat{x_k} \cdots x_n$ denotes the product $x_1 \cdots x_n$ that omits x_k. Similarly, if $I_1, \dots, I_{n+1}$ are ideals of R, then $I_1 \cdots \widehat{I_k} \cdots I_{n+1}$ denotes the product $I_1, \dots, I_{n+1}$ that omits I_k. We start with the following lemmas.

Lemma 5.1. *Let R be a commutative ring. Suppose there are ideals $I_1, \dots, I_{n+1}$ of R such that $I_1 \cdots . I_{n+1} = \{0\}$ and no product of n of the I_j's is equal to $\{0\}$. Then there are finitely generated ideals $J_1, \dots, J_{n+1}$ of R such that $J_1 \cdots J_{n+1} = \{0\}$ and no product of n of the J_i's is equal to $\{0\}$.*

Proof. Suppose there are ideals $I_1, \dots, I_{n+1}$ of R such that $I_1 \cdots . I_{n+1} = \{0\}$ and no product of n of the I_j's is equal to $\{0\}$.

Let $j \in \{1, \dots, n+1\}$. Since $\prod_{i=1, i\neq j}^{n+1} I_i \neq \{0\}$ for all $i \neq j$, there exist $a_{i,j} \in I_i$ such that $\prod_{i=1, i\neq j}^{n+1} a_{i,j} \neq \{0\}$. Let $J_j = (a_{1,j}, \dots, \widehat{a_{j,j}}, \dots, a_{n+1,j})$ the ideal generated by $\{a_{i,j}, i \neq j, i = 1, \dots, n+1\}$. Since $J_j \subseteq I_j$, we have $J_1 \cdots J_{n+1} = \{0\}$. Thus, $\prod_{i=1, i\neq j}^{n+1} J_i \neq \{0\}$, for every $j \in \{1, \dots, n+1\}$, as desired. □

Lemma 5.2. *Suppose that in any ring $\{0\}$ is a strongly n-absorbing ideal if and only if $\{0\}$ is an n-absorbing ideal. Then every n-absorbing ideal in an arbitrary ring R is a strongly n-absorbing ideal of R.*

Proof. Suppose I is n-absorbing ideal in a ring A and let the canonical homomorphism $f : R \to R/I$. Then $\{0\}$ is an n-absorbing ideal of $A' = A/I$ by [2, Theorem 4.2] and thus $\{0\}$ is a strongly n-absorbing ideal of A'. Let $I_1, \dots, I_{n+1}$ are ideals of A such that $\prod_{i=1}^{n+1} I_i \subset I$, then $\prod_{i=1}^{n+1} f(I_i) = \{0\}$. Since $\{0\}$ is a strongly n-absorbing ideal of A', there exist $j \in \{1, \dots, n+1\}$ such that $\prod_{i=1, i\neq j}^{n+1} f(I_i) = \{0\}$ and so $\prod_{i=1, i\neq j}^{n+1} I_i \subset I$. Therefore, I is a strongly n-absorbing ideal of A. □

Lemma 5.3. *Let R be a commutative u-ring such that $\{0\}$ is an n-absorbing ideal. Then $\{0\}$ is a strongly n-absorbing of R.*

Proof. Let $I_1, \dots, I_{n+1}$ be ideals of R such that $I_1 \cdots I_{n+1} = \{0\}$. Assume that there is no product of n ideals of the I_j's equals to zero. By Lemma 5.2, there are finitely generated ideals $J_1, \dots, J_{n+1}$ of R such that $J_1 \cdots J_{n+1} = \{0\}$ and no product of n of the J_i's equals to $\{0\}$. Let n_j be the minimal number of generators for J_j, and $\varphi(J_1, \dots, J_{n+1}) = \sum_{i=1}^{n+1} n_j$. It is clear that $\varphi(J_1, \dots, J_{n+1}) \in \{n+1, \dots, n(n+1)\}$.

We will show by induction that there exists a product of n ideals of the J_i's equals to zero, which is the desired contradiction.

Suppose that $\varphi(J_1, \dots, J_{n+1}) = \sum_{i=1}^{n+1} n_j = n+1$. Then for every $j = 1, \dots, n+1$, there exists an element $a_j \in R$ such that $J_j = Ra_j$. Hence, $J_1 \cdots .J_{n+1} = \{0\}$. Since $\{0\}$ is an n-absorbing ideal of R, there exists one product $a_1 \cdots \widehat{a_k} \cdots a_{n+1} = \{0\}$ and hence $J_1 \cdots \widehat{J_k} \cdots J_{n+1} = \{0\}$.

Now, assume that whenever $L_1 L_2 \cdots L_{n+1} = \{0\}$ for some ideals $L_1, \dots, L_{n+1}$ of R and $\varphi(L_1, \dots, L_{n+1}) < \varphi(J_1, \dots, J_{n+1})$, there exists a $k \in \{1, \dots, n+1\}$ such that $L_1 \cdots \widehat{L_k} \cdots L_{n+1} = \{0\}$. Since $\sum_{j=1}^{n+1} n_j > n+1$, without loss of generality, suppose $n_1 > 1$, and let $a_1 \in J_1$. Then $a_1 J_2 \cdots J_{n+1} = \{0\}$. Let $L_1 = Ra_1$, and for $j \geq 2$, let $L_j = J_j$. Hence $L_1 \cdots L_{n+1} = \{0\}$ and $\varphi(L_1, \dots, L_{n+1}) = 1 + \sum_{k=2}^{n+1} n_k < \varphi(J_1, \dots, J_{n+1})$. By induction there exists some $j \in \{2, \dots, n+1\}$ such that $L_1 J_2 \cdots \widehat{J_j} \cdots J_{n+1} = \{0\}$. Since $J_2 \cdots .J_{n+1} \neq \{0\}$ by hypothesis, we have $a_1 \in \mathrm{ann}(Q_j)$, where $Q_j = J_2 \cdots \widehat{J_j} \cdots J_{n+1}$. Thus, $J_1 \subset \bigcup_{i=1}^{n+1} \mathrm{ann}(Q_j)$. Since R is a u-ring, there exists $j \in \{1, \dots, n+1\}$ such that $J_1 \subset \mathrm{ann}(Q_j)$. Thus, $J_1 \dots \widehat{J_j} \cdots J_{n+1} = \{0\}$, a contradiction. Therefore, there exists $j \in \{1, \dots, n+1\}$ such that $I_1 \cdots \widehat{I_j} \cdots I_{n+1}$ equals to zero. Hence $\{0\}$ is a strongly n-absorbing of R. □

Theorem 5.4. *Let R be a commutative u-ring. Then R satisfies Conjecture one, that is every n-absorbing ideal of R is a strongly n-absorbing ideal of R.*

Proof. Let R be a commutative u-ring. Suppose that I is a proper n-absorbing ideal of R. Then the quotient ring R/I is a u-ring by [30, Proposition 1.3] and $\{0\}$ is an n-absorbing ideal of R/I. Therefore, $\{0\}$ is a strongly n-absorbing of R/I by Lemma 5.3. Hence I is a strongly n-absorbing ideal of R. □

We recall from [30] that a ring A is called a *um-ring* if whenever an R-module equal to a finite union of submodules must be equal to one of them.

Remark 5.5. Let R be a commutative ring and assume that R contains an infinite set S such that $x - y$ is a unit for all $x \neq y$ in S. Then R is a *um*-ring by [30, Proposition 1.7]. It is shown [30, Theorem 2.3] that a ring R is a *um*-ring if and only if R/M is infinite for every maximal ideal M of R. It is shown [30, Theorem 2.6] that a ring R is an u-ring if and only if R/M is infinite or R_M is a Bezout ring for every maximal ideal M of R. Hence in view of [30, Theorem 2.3] and [30, Theorem 2.6], we conclude that every *um*-ring is a u-ring. The converse is not true, for let $R = \mathbb{Z}$. Then R is a u-ring. Since R/M is finite for every maximal ideal M of R, we conclude that R is not a *um*-ring.

In view of Remark 5.5, we have the following result.

Theorem 5.6. *Let R be a um-ring. Then R is a u-ring.*

The proof of the following result is similar to the proof of [30, Proposition 1.7].

Theorem 5.7. *Let R be a commutative ring with $1 \neq 0$, n be a positive integer, and I be a proper ideal of R. Suppose that R contains an infinite set S such that $x - y$ is a unit for all $x \neq y$ in S. Then R is a u-ring, and hence I is a strongly n-absorbing of R if and only if I is an n-absorbing ideal of R.*

Proof. Suppose that R contains an infinite set S such that $x - y$ is a unit for all $x \neq y$ in S. We show that R is a u-ring. Deny. Let I be an ideal of R and $p \geq 1$ be an integer such that $I \subset \bigcup_{i=1}^{p} I_i$, and suppose that for every $i \in \{1, \ldots, p\}$, we have $I \nsubseteq I_i$. We may assume that for each $i \in \{1, \ldots, p\}$, we have $I \nsubseteq \bigcup_{j \neq i} I_j$. Hence for each $1 \leq i \leq 2$, there exists $a_i \in I$ such that $a_i \notin \bigcup_{j \neq i} I_j$. Consider the set $H = \{a_1 + xa_2 \,|\, x \in S\}$. Then for every $x \in S$, we have $a_1 + xa_2 \in I$ and $a_1 + xa_2 \notin I_2$. Since $H \subseteq I$ and $H \cap I_2 = \emptyset$, we have $H \subset \bigcup_{j \neq 2} I_j$. Since H is infinite, there exist $x_1 \neq x_2$ in S such that $a_1 + x_1 a_2$ and $a_1 + x_2 a_2 \in I_i$ for some $i \neq 2$. Hence $(x_1 - x_2)a_2 \in I_i$, and thus $a_2 \in I_i$, which is a contradiction. Thus, R is a u-ring. □

Remark 5.8. One can give an alternative proof of Theorem 5.7. Note that since R contains an infinite set S such that $x - y$ is a unit for all $x \neq y$ in S, we conclude that R is a *um*-ring by [30, Proposition 1.7]. Hence R is a u-ring by Theorem 5.6.

Theorem 5.9. *Let A be a u-domain with quotient field K, M be a K-vector space, and $R = A(+)M$. Then Conjecture one holds in R.*

Proof. Since A satisfies Conjecture one by Theorem 5.4, we conclude that R satisfies Conjecture one by Theorem 3.4. □

The following is an example of a ring that is not a u-ring but it satisfies Conjecture one.

Example 5.10. Let $R = \mathbb{Z}_3(+)\mathbb{Z}_3[X]$. Then R satisfies Conjecture one by Theorem 5.9. It is clear that $M = \{0\}(+)\mathbb{Z}_3[X]$ is the only maximum ideal of R. Since neither R/M is infinite (note that $R/M \cong \mathbb{Z}_3$) nor R_M (note that $R_M = R$) is a Bezout ring, we conclude that R is not a u-ring by [30, Theorem 2.6]. Note that R is not a um-ring by Theorem 5.6.

Theorem 5.11. *Let A be a commutative um-ring, M be an A-module, and $R = A(+)M$. Then Conjecture one holds in R.*

Proof. Let H be a maximal ideal of R. Then $H = L(+)M$ for some maximal ideal L of A. Since $R/H \cong A/L$ and A is a um-ring, we conclude that A/L is infinite, and thus R/H is infinite. Hence R is a um-ring by [30, Theorem 2.3]. Thus, R is a u-ring by Theorem 5.6. Hence R satisfies Conjecture one by Theorem 5.4. □

6. (m, n)-Closed Ideals in Trivial Ring Extension

Let R be a commutative ring with $1 \neq 0$. We recall from [3] that a proper ideal I of R is called an (m, n)-*closed* ideal if $x^m \in I$ for $x \in R$ implies $x^n \in I$.

Theorem 6.1. *Let A be a ring, M be an R-module, and $R = A(+)M$. Suppose that $J = I(+)N$ is a proper ideal of R, where I is a proper ideal of A and N is a submodule of M such that $IM \subseteq N$. If I is an (m, n)-closed ideal of A for some integers $0 < n < m$, then J is an $(m, n+1)$-closed ideal of R.*

Proof. Suppose that I is an (m, n)-closed ideal of A for some integers $0 < n < m$. Let $x = (a, c) \in R$ and suppose that $x^m = (a^m, ma^{m-1}c) \in J$. Since I is an (m, n)-closed ideal of A, we conclude that $(a^{n+1}, (n+1)a^n c) = x^{n+1} \in J$. Thus J is an $(m, n+1)$-closed ideal of R. □

In view of Theorem 6.1, the following is an example of an $(3, 2)$-closed ideal I of Z but the proper ideal $J = I(+)I$ of $R = Z(+)Z$ is not an $(3, 2)$-closed ideal of R.

Example 6.2. Let $R = Z(+)Z$, $p \neq 2$ be a positive prime number of Z, $I = p^4 Z$ a proper ideal of Z, and $J = I(+)I$. Then J is a proper ideal of R and I is an $(3, 2)$-closed ideal of Z by [3, Corollary 3.3]. Let $x = (p^2, p) \in R$. Then $x^3 = (p^6, 3p^5) \in J$. Since $p \neq 2$, we have $x^2 = (p^4, 2p^3) \notin J$.

Lemma 6.3. *Let A be a ring, M be an R-module, and $R = A(+)M$. Suppose that $J = I(+)N$ is a proper ideal of R, where I is an (m, n)-closed ideal of A for some integers $0 < n < m$, and N is a submodule of M such that $IM \subseteq N$. Let $x = (a, c) \in R$ for some $a \in A$ and $c \in M$. Then $x^m \in J$ if and only if $a^m \in I$.*

Proof. Suppose that $x^m = (a^m, ma^{m-1}c) \in J$. Then it is clear that $a^m \in I$.

Conversely, suppose that $a^m \in I$. Since I is an (m, n)-closed ideal of R, $a^n \in I$. Since $n \leq m - 1$, we conclude that $a^{m-1} \in I$. Since $IM \subseteq N$ and $a^{m-1} \in I$, we conclude that $x^m = (a^m, ma^{m-1}c) \in J$. □

Theorem 6.4. *Let A be a ring, M be an R-module, and $R = A(+)M$. Suppose that $J = I(+)N$ is a proper ideal of R, where I is a proper ideal of A and N is a submodule of M such that $IM \subseteq N$. Let $0 < n < m$ be integers. The following statements are equivalent*:

(1) *J is an (m, n)-closed ideal of R.*
(2) *I is an (m, n)-closed ideal of A and whenever $a^m \in I$ for some $a \in A$ implies $na^{n-1}M \subseteq N$.*

Proof. (1) ⇒ (2). Suppose that J is an (m, n)-closed ideal of R. Then it is clear that I is an (m, n)-closed ideal of A. Assume that $a^m \in I$ for some $a \in A$. Let $c \in M$ and $x = (a, c)$. Since $a^m \in I$, we have $x^m \in R$ by Lemma 6.3. Since J is an (m, n)-closed ideal of R, we conclude that $x^n = (a^n, na^{n-1}c) \in R$. Thus, $na^{n-1}M \subseteq N$.

(2) ⇒ (1). Suppose that I is an (m, n)-closed ideal of A and whenever $a^m \in I$ for some $a \in A$ implies $na^{n-1}M \subseteq N$. Let $x = (a, c) \in R$ for some $a \in A$ and $c \in M$ and suppose that $x^m = (a^m, ma^{m-1}c) \in J$. Since $a^m \in I$ and I is an (m, n)-closed ideal of A, we conclude that $a^n \in A$ and $na^{n-1}c \in N$. Thus, $x^n = (a^n, na^{n-1}c) \in J$. Hence J is an (m, n)-closed ideal of R. □

Theorem 6.5. *Let A be a ring, M be an R-module, m and n integers with $1 \leq n < m$, I be a proper ideal of A, and $R = A(+)M$. Suppose that char$(A) \mid n$. Then the following statements are equivalent*:

(1) *$J = I(+)N$ is an (m, n)-closed ideal of R for every submodule N of M where $IM \subseteq N$.*
(2) *I is an (m, n)-closed ideal of A.*

Proof. (1) ⇒ (2). It is clear by Theorem 6.4.

(2) ⇒ (1). Let N be a submodule of M such that $IM \subseteq N$. Since char$(A) \mid n$, we conclude that whenever $a^m \in I$ for some $a \in A$ implies $na^{n-1}M = 0_M \subseteq N$, where 0_m is the additive identity of M. Thus, $J = I(+)N$ is an (m, n)-closed ideal of R by Theorem 6.4. □

Theorem 6.6. *Let D be an integral domain, $R = D(+)D$, m and n integers with $1 \leq n < m$, and $I = p^kD$, where p is a prime element of D and k is a positive integer. Suppose that $m > k$ and* char$(D) \neq n$. *Then the following statements are equivalent*:

(1) *$J = I(+)p^iD$ is an (m, n)-closed ideal of R for some integer $i \geq 1$.*

(2) *One of the following three cases must hold*:

   (a) $k < n < m$ *and* $i \leq k$.
   (b) $n = k$, *and* $1 \leq i < k$.
   (c) $n = i = k$, *and* $p \,|\, k \cdot 1_D$ (*in* D), *where* 1_D *is the identity of* D.

Proof. (1) ⇒ (2). Suppose that $J = I(+)p^iD$ is an (m,n)-closed ideal of R for some integer $i \geq 1$. Since J is an ideal of R, we conclude that $I \subseteq p^iD$. Hence $i \leq k$. Since $J = I(+)p^iD$ is an (m,n)-closed ideal of R, we conclude that I is an (m,n)-closed ideal of D and whenever $a^m \in I$ for some $a \in D$ implies $na^{n-1}D \subseteq p^iD$ by Theorem 6.4. Since $m > k$, $p^m \in I$ and hence $p^n \in I$ and $np^{n-1}D \subseteq p^iD$. In particular, $np^{n-1} \in p^iD$. Since $p^n \in I$, we conclude that $n \geq k$. Suppose that $n = k$. Then $np^{n-1} = kp^{k-1} \in p^iD$ if and only if either $1 \leq i < k$ or $i = k$ and $p \,|\, k \cdot 1_D$.

(2) ⇒ (1). In view of proof (1) ⇒ (2) above, one can easily verify that if (a) or (b) or (c) holds, then I is an (m,n)-closed ideal of D and whenever $a^m \in I$ for some $a \in D$ implies $na^{n-1}D \subseteq p^iD$. Hence J is an (m,n)-closed ideal of R by Theorem 6.4. □

Definition 6.7. Let p be a prime element of an integral domain D. Suppose that $p^w \,|\, d$ for some $d \in D$ and a positive integer w but $p^{w+1} \nmid d$. Then we write $p^w \,\|\, d$.

Theorem 6.8. *Let D be an integral domain, $R = D(+)D$, m and n integers with $1 \leq n < m$, and $I = p^kD$, where p is a prime element of D and k is a positive integer. Suppose that $m < k$ and $char(D) \neq n$. Let $v = \lceil \frac{k}{m} \rceil$ and $u = \lceil \frac{k}{v} \rceil$. Then the following statements are equivalent*:

(1) $J = I(+)p^iD$ *is an* (m,n)-*closed ideal of* R *for some integer* $i \geq 1$.
(2) *One of the following three cases must hold*:

   (a) $u < n < m$ *and* $i \leq k$.
   (b) $u = n, p \nmid n \cdot 1_D$ (*in* D), *and* $i \leq v(n-1) < k$.
   (c) $u = n$, $p^w \,\|\, n \cdot 1_D$ (*in* D), *and* $i \leq \min\{v(n-1) + w, k\}$.

Proof. (1) ⇒ (2). Suppose that $J = I(+)p^iD$ is an (m,n)-closed ideal of R for some integer $i \geq 1$. Since J is an ideal of R, we conclude that $I \subseteq p^iD$. Hence $i \leq k$. It is clear that $v = \lceil \frac{k}{m} \rceil$ is the smallest positive integer where $(p^v)^m \in I$. Also, it is clear that u is the smallest positive integer where $(p^v)^u \in I$. Since $J = I(+)p^iD$ is an (m,n)-closed ideal of R and $1 \leq n < m$, we conclude that $u \leq n < m$. Since $J = I(+)p^iD$ is an (m,n)-closed ideal of R, we conclude that I is an (m,n)-closed ideal of D and whenever $a^m \in I$ for some $a \in D$ implies $na^{n-1}D \subseteq p^iD$ by Theorem 6.4. Hence since $(p^v)^m \in I$, we conclude that $n(p^v)^{n-1} \in p^iD$ by Theorem 6.4. If $u < n < m$, then $u \leq n-1$ and thus $n(p^v)^{n-1} \in p^kD = I$ (note that $(p^v)^u \in I$) and $i \leq k$. Suppose that $n = u$ and $p \nmid n \cdot 1_D$ (in D). Since u is the smallest positive integer where $(p^v)^u \in I$ and $p \nmid n \cdot 1_D$, we conclude that

$v(n-1) < k$ and $n(p^v)^{n-1} \in p^i D$ if and only if $i \le v(n-1) < k$. Suppose that $u = n$ and $p^w \,\|\, n \cdot 1_D$ (in D). Since $i \le q$, we conclude that $n(p^v)^{n-1} \in p^i D$ if and only if $i \le \min\{v(n-1)+w, k\}$.

(2) $\Rightarrow$ (1). In view of proof (1) $\Rightarrow$ (2) above, one can easily verify that if (a) or (b) or (c) holds, then I is an (m,n)-closed ideal of D and whenever $a^m \in I$ for some $a \in D$ implies $na^{n-1}D \subseteq p^i D$. Hence J is an (m,n)-closed ideal of R by Theorem 6.4. □

Let R be an integral domain, $I = p^k R$, where p is a prime element of R and k is a positive integer, and m and n be fixed positive integers with $1 \le n < m$. The authors in [3, Theorem 3.1] determined the set $\{k \in \mathbb{N} \mid p^k R \text{ is } (m,n)\text{-closed}\}$. We recall the following result.

Theorem 6.9 ([3, Theorem 3,1]). *Let D be an integral domain, m and n integers with $1 \le n < m$, and $I = p^k D$, where p is a prime element of D and k is a positive integer. Then the following statements are equivalent*:

(1) *I is an (m,n)-closed ideal of D.*
(2) *If $m = bn + c$ for integers b and c with $b \ge 2$ and $0 \le c \le n-1$, then $k \in \{1, \dots, n\}$. If $m = n + c$ for an integer c with $1 \le c \le n-1$, then $k \in \bigcup_{h=1}^{n} \{mi + h \mid i \in \mathbb{Z} \text{ and } 0 \le ic \le n-h\}$.*

In view of Theorems 6.6, 6.8 and 6.9, we have the following result.

Theorem 6.10. *Let D be an integral domain, $R = D(+)D$, m and n integers with $1 \le n < m$, and $I = p^k D$, where p is a prime element of D and k is a positive integer. Suppose that $\mathrm{char}(D) \ne n$. Then the following statements are equivalent*:

(1) *$J = I(+)p^i D$ is an (m,n)-closed ideal of R for some integer $i \ge 1$.*
(2) *If $m = bn + c$ for integers b and c with $b \ge 2$ and $0 \le c \le n-1$, then $k \in \{1, \dots, n\}$ and one of the following three cases must hold*:
   (a) *$k < n < m$ and $i \le k$.*
   (b) *$n = k$, and $1 \le i < k$.*
   (c) *$n = i = k$, and $p \mid k \cdot 1_D$ (in D), where 1_D is the identity of D.*

   If $m = n + c$ for an integer c with $1 \le c \le n-1$, then $k \in \bigcup_{h=1}^{n} \{mi + h \mid i \in \mathbb{Z}$ and $0 \le ic \le n-h\}$ and one of the following three cases must hold:
   Let $v = \lceil \frac{k}{m} \rceil$ and $u = \lceil \frac{k}{v} \rceil$. Then
   (a) *$u < n < m$ and $i \le k$.*
   (b) *$u = n, p \nmid n \cdot 1_D$ (in D), and $i \le v(n-1) < k$.*
   (c) *$u = n$, $p^w \| n \cdot 1_D$ (in D), and $i \le \min\{v(n-1)+w, k\}$.*

Proof. (1) $\Rightarrow$ (2). Suppose that $J = I(+)p^i D$ is an (m,n)-closed ideal of R for some integer $i \ge 1$. Then I is an (m,n)-closed ideal of D by Theorem 6.4. Suppose that $m = bn+c$ for integers b and c with $b \ge 2$ and $0 \le c \le n-1$. Then $k \in \{1, \dots, n\}$ by Theorem 6.9. Hence $m > k$. Thus we are done by Theorem 6.6. Suppose that

$m = n + c$ for an integer c with $1 \le c \le n - 1$. Then $k \in \bigcup_{h=1}^{n}\{mi + h \,|\, i \in \mathbb{Z}$ and $0 \le ic \le n - h\}$ by Theorem 6.9. Thus, $m < k$. Hence we are done by Theorem 6.8.

(2) ⇒ (1). Suppose that $k \in \{1, \ldots, n\}$ and (a) or (b) or (c) holds. Since $m > k$, we are done by Theorem 6.6. Suppose that $m = n + c$ for an integer c with $1 \le c \le n - 1$ and $k \in \bigcup_{h=1}^{n}\{mi + h \,|\, i \in \mathbb{Z}$ and $0 \le ic \le n - h\}$ and (a) or (b) or (c) holds. Since $m < k$, we are done by Theorem 6.8. □

In view of Theorems 6.1 and 6.9, we have the following result.

Theorem 6.11. *Let D be an integral domain, $I = p^k D$, where p is a prime element of D and k is a positive integer, M be a D-module, $R = D(+)M$, $J = I(+)N$ is a proper ideal of R, where N is a submodule of M such that $IM \subseteq N$, and m and n integers with $1 \le n < m$. Then the following statements are equivalent*:

(1) *I is an (m, n)-closed ideal of D and J is an $(m, n+1)$-closed ideal of R.*
(2) *If $m = bn + c$ for integers b and c with $b \ge 2$ and $0 \le c \le n - 1$, then $k \in \{1, \ldots, n\}$. If $m = n + c$ for an integer c with $1 \le c \le n - 1$, then $k \in \bigcup_{h=1}^{n}\{mi + h \,|\, i \in \mathbb{Z}$ and $0 \le ic \le n - h\}$.*

Proof. (1) ⇒ (2). Suppose that I is an (m, n)-closed ideal of D and J is an $(m, n+1)$-closed ideal of R. Since I is an (m, n)-closed ideal of D, we are done by Theorem 6.9.

(2) ⇒ (1). By Theorem 6.9, I is an (m, n)-closed ideal of D. Hence J is an $(m, n+1)$-closed ideal of R by Theorem 6.1. □

Theorem 6.12. *Let A be an integral domain with quotient field K, M be a K-vector space, and $R = A(+)M$. Then the following statements are equivalent*:

(1) *Every proper ideal of A is an (m, n)-closed ideal of A for some integers $1 \le n < m$.*
(2) *Every proper ideal of R is an (m, n)-closed ideal of R for some integers $1 \le n < m$.*

Proof. (1) ⇒ (2). Suppose that every proper ideal of A is an (m, n)-closed ideal of A for some integers $1 \le n < m$. Let J be an ideal of R. Since M is a divisible A-module, we have $J = I(+)M$ for some proper ideal I of A or $J = \{0\}(+)N$ for some A-submodule N of M by ([1, Corollary 3.4]). Suppose that $J = I(+)M$ for some proper ideal I. Since I is an (m, n)-closed ideal of A for some integers $1 \le n < m$, it is clear that $J = I(+)M$ is an (m, n)-closed ideal of R. Suppose that $J = \{0\}(+)N$ for some A-submodule N of M. Since A is an integral domain, we have $J = \{0\}(+)N$ is an $(m, 2)$-closed ideal of R for every integer $m \ge 3$. Hence every proper ideal of R is an (m, n)-closed ideal of R for some integers $1 \le n < m$.

(2) ⇒ (1). It is clear. □

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